

Final exam Linear Algebra II

Thursday 09/04/2026, 15:00–17:00

1 (2 + 3 + 2 + 2 = 9 pts)

linear transformations, dimension theorem

Consider the \mathbb{R} -vector space P_2 of real polynomials of degree ≤ 2 .

(a) Let T be the mapping from P_2 to P_2 defined by

$$(T(p))(x) := 3x^2p\left(\frac{1}{x}\right) - 3p(x).$$

Show that T is a linear transformation.

(b) Find the matrix representation of T with respect to the basis $(1, x, 3x^2 - 3)$.

(c) Prove that the set

$$\mathcal{S} = \{p \in P_2 \mid 3x^2p\left(\frac{1}{x}\right) - 3p(x) = 0\}$$

is a subspace of P_2 .

(d) Find the dimension of \mathcal{S} .

2 (3 + 3 + 3 = 9 pts)

inner product, orthonormal basis, adjoint

We continue with P_2 and $T: P_2 \rightarrow P_2$ as described in exercise 1. Define $\langle \cdot, \cdot \rangle: P_2 \times P_2 \rightarrow \mathbb{R}$ by

$$\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2 \rangle := 3(a_0 + a_2)(b_0 + b_2) + a_1b_1 + a_2b_2.$$

(a) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on P_2 .

(b) Change the basis described in 1(b) into an orthonormal basis for P_2 .

(c) Compute $T^{\text{adj}}(3x^2 - 3)$.

3 (2 + 5 + 2 = 9 pts)

norms, orthogonal projections, least squares

Consider the \mathbb{R} -inner product space V consisting of all continuous functions $f: [0, 1] \rightarrow \mathbb{R}$. Here V is equipped with the inner product $\langle f, g \rangle := \int_0^1 f(t)g(t) dt$. The subspace $W \subset V$ consists of all real polynomials $p(t)$ of degree ≤ 2 , with the property $p(\frac{1}{2}) = 0$. The function $v \in V$ is defined by $v(t) = |2t - 1|$.

- Compute the norm $\|v\|$.
- Show that $t \mapsto \frac{5}{4}(2t - 1)^2$ is the orthogonal projection of v onto the subspace $W \subset V$.
- What is the minimal value of $\|w - v\|$ taken over all $w \in W$?

4 (2 + 5 + 2 = 9 pts)

singular value decomposition

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

- Compute the singular values of A .
- Find a singular value decomposition of A .
- Determine the distance $d(A, M_1)$ and compute a best rank-1 approximation of A . (Recall that here $M_1 = \{S \in \mathbb{R}^{2 \times 3} : \text{rank}(S) \leq 1\}$).

4 pts free